DEFERRED CORRECTIONS

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"The root of the matter is that the greatest stimulus
of scientific discovery are its practical applications."
Lewis Fry Richardson (1908)

1. Introduction

In this paper we give a review of developments in deferred corrections and related methods from the beginning to today. By deferred corrections we mean approximate methods to solve differential and integral equations that use a basic low order method and successive corrections to achieve a higher order of convergence with respect to the integration step. This approach has two main advantages over direct high order methods: (a) it can use the basic stability and convergence results for the low order method at all stages; (b) for boundary value problems the basic problem always have the same bandwith and it uses a fixed mesh. An added bonus is that the techniques are intimately related to global error estimation, so that adaptive methods, both in the order of the method and in the mesh distribution, arise naturally.

Leslie Fox (1918-1992) is the father of difference/deferred corrections [19]. He coined both names as we can see in his 1962 book with D. F. Mayers and R. A. Buckinghan. It is clear in his early writings the influence of human computer needs. As explained below, although large distributed computer networks have made possible to solve problems of increasing size and complexity in reasonable amounts of wall clock time, we are not very close to exhausting our thirst for even larger and more complex calculations, so that the need for sophisticated and efficient algorithms is still very present, as opposite to simple, brute force ones.

By now, it has been abundantly shown that for those problems in which high order methods are appropriate, they are considerably more efficient than low order ones, so that is another reason to pursue this approach to high order methods.

Although there has been considerable activity in this field in the past 50 years, one purpose of this work is to call the attention of those that are not conversant with the subject and that can benefit from it. In the words of Leslie Fox [23]: "This idea (difference corrections) does not seem to have penetrated deeply into the literature of automatic computation ... Certainly

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we have to do some differencing, involving extra programming, extra space, and [there are] some difficulties in automatic inspection of differences, but machines are getting larger and programming easier (or so everybody tells me), and if we are concerned with accuracy, as we certainly should be, I should have thought that something like this was essential."

This was said in 1963, and although substantial progress have been made, there is still room for more appreciation of the power and efficiency of these ideas.

2. Difference corrections

2.1. **The very early days of hand computing.** It is not casual that we quote L. F. Richardson at the beginning of this paper. He is one of the founders of modern scientific computing, together with Southwell and Fox, starting very early in the XX Century and, coincidentally, Richardson's extrapolation is also a method to improve the order of a discretization method. Until the 1940's, numerical computing was performed by humans, either by hand or with the aid of mechanical calculators and therefore economy in the number of operations necessary for a given task was a must. In the 1940's, fueled by second world war needs, stored program computers of practical value were first developed and the real revolution in computing started. Fox is right in the middle of this transition and that shows in his use and understanding of tables of differences. However, both he and Mayers recognized the difficulty of automating the difference correction process in the way they conceived it.

2.2. **Leslie Fox and difference tables.** A difference table can be represented by a matrix where the first column corresponds to the evaluation of a function y of one variable at successive points of a uniform mesh, while the successive columns correspond to the first differences of the previous one, i.e., the higher order differences of $y(x)$.

By judicious use of Taylor expansions one gets for the approximation by centered differences of the second derivative

$$h^2 y_i^{(2)} = \delta_i^2 - \frac{1}{12}\delta_i^4 + \frac{1}{90}\delta_i^6 + \cdots ,$$

or, in terms of derivatives

$$h^{-2}\delta_i^2 = y_i^{(2)} + \frac{h^2}{12}y_i^{(4)} + \frac{h^4}{360}y_i^{(6)} + \mathcal{O}(h^6). \tag{2.1}$$

In his association with Southwell and with an obvious abound-ant knowledge and experience with practical calculations, Fox systematically combines Southwell's relaxation methods to solve iteratively the systems of linear equations that result in the discretization of linear boundary value problems, with the approximation of derivatives by "infinite series of differences" in a deferred

y	$\triangle y$	$\triangle^2 y$	...	$\triangle^n fy$
y_1	$y_2 - y_1$	$y_3 - 2y_2 + y_1$		
y_2	$y_3 - y_2$	etc		
...	...			
y_{n+1}				

TABLE 1. Table of differences of a function of one variable

correction manner. We believe that this terminology is an attempt to validate and give a mathematical veneer to a technique that clearly works well in practice but lacks a theory of convergence and where only a limited number of terms are ever used in practice. This way of thinking though, seems to have been an obstacle for the various attempts to obtain a mathematical proof of convergence for the method.

In fact, there seems to be no clear recognition that this infinite series of differences or their counterparts in terms of derivatives are not to be thought in general as convergent series, but rather as asymptotic ones, where by using a limited number of terms any accuracy can be obtained by reducing the step size. Again, these details did not hamper the practical application of the technique, where Fox had a prescient modern view that pervaded the next wave of developments.

3. Deferred correction

3.1. **Digital computer era.** In the early 1960's, both P. Henrici and M. Lees came up with proofs of convergence of a one step difference correction for a simple scalar two point boundary value problem. Henrici's 'proof' is contained in Problem 21, p. 387 of his book [29], while Lees paper appeared in a conference proceedings [38]. It was this last paper and Gene Golub's interest in it that brought the author to the subject in 1964.

He started to write his Doctoral thesis remotely for the University of Buenos Aires, while at Stanford and then at the Mathematics Research Center of the University of Wisconsin in Madison. His nominal supervisor in Buenos Aires was none other than Pedro Zadunaisky, which will come later with a bifurcated version of deferred corrections that he called "the pseudo-problem".

Zadunaisky, an astronomer, was interested in initial value problems for ODE's and most specially in carefully monitoring the errors of integration of orbits [64, 65]. There was a departure in his developments from VP's approach that eventually led to Stetter et al's defect correction method. More details follow below.

3.2. **Early theory.** A brief abstract formulation of deferred corrections starts with a continuous problem $F(y)$ (i.e., an ordinary of partial differential equation or integral equation and its boundary conditions) and its discretization $F_h(y_h)$, where h is a small parameter related to the step size. We use the mapping $y_h = \phi_h y$ to indicate the discretization of the continuous function $y(x)$. We assume that the continuous problem has at least one isolated solution y. Likewise, let y_h be the corresponding solution of the discrete approximation.

The quantity $d_h = F_h(\phi_h y)$ is called the *local truncation error* or *defect*. If $d_h = \mathcal{O}(h^p)$, we say that the method is *consistent of order p*. The method is *convergent of order p* if $lim_{h\downarrow 0}\|y_h - \phi_h y\| = 0$, where the quantity $e = \phi_h y - y_h$ is called the *global error*.

Lax and Richtmyer equivalence theorem [36] says that a linear difference scheme is convergent iff is consistent and stable. Although the original concept was carefully crafted for time dependent linear initial value problems, it extends naturally to boundary value and nonlinear problems. A discretization of a linear problem is *stable* if

$$\|y_1 - y_2\| < c\|F_h(y_1) - F_h(y_2)\|,$$

where c is independent of h. This is also valid for nonlinear problems. For differentiable ones it can be replaced by a linearized version, where the Frechet derivative (Jacobian) $F_h^{'}$ takes the place of F_h. Observe that this connects loosely with the theory of monotone operators of Zarantonello, Minty, Petryshyn and Browder, coercive operators and the abstract implicit function theorem for nonlinear mappings. **In simple words, the method is stable if the inverse of its Frechet derivative is bounded, independent of the step** h.

3.3. **Asymptotic expansions.** As was indicated above, finite difference approximations to derivatives of smooth functions lead naturally to asymptotic expansions. Thus, we have for instance the expansion in (2.1) for the second derivative approximation by a centered second order difference on a uniform mesh. Observe that this expansion contains only even powers of the step.

Gragg for initial value problems for ordinary differential equations [25] and then Stetter for general functional equations [60], proved that under certain conditions an asymptotic expansion for the local error leads to one for the global error. These results were used to prove the order of convergence of iterated Richardson extrapolations [58].

This relatively simple technique is predicated on the idea that if the global error has an asymptotic expansion, it is possible to improve the order of a basic method by combining discrete solutions corresponding to different values of the step. The classical formula

$$Ry_h = \frac{4y_{h/2} - y_h}{3}$$

produces an $\mathcal{O}(h^4)$ method by combining the solutions of a basic order $\mathcal{O}(h^2)$ method for two values of the step size, h and $h/2$. The assumption is that

$$\phi_h y - y_h = Ch^2 + \mathcal{O}(h^4).$$

Observe that the improved solution is obtained only on the coarsest grid.

The term asymptotic expansion refers to divergent power series that are function approximations [62]. A formal definition:

Definition 1. A function $f(x)$ defined at $x = 0$ is said to be approximated asymptotically by the power series $\Sigma_{r=0}^{\infty} a_r x^r$ at $x = 0$ if for all $m \geq 0$

$$f(x) = \Sigma_{r=0}^{m} a_r x^r + \mathcal{O}(x^{m+1}).$$

An obviously modified definition arises if the relationship is only available for a finite number of terms. A natural way to obtain an asymptotic approximation to a differentiable function is Taylor's formula. Properties and more general statements can be found in Wasow's book [62].

In (2.1) we saw an example of asymptotic expansion associated with the second order symmetric difference. Thus, if we consider the two-point boundary value problem

$$y" = f(x, y), \quad y(0) = a, \quad y(1) = b \tag{3.1}$$

and its discretization on an uniform mesh $x_i = ih,\ h = 1/n$

$$h^{-2}\delta^2 y_{hi} = f(x_i, y_{hi}),\ i = 1, ..., n-1 \quad y_0 = a, \quad y_n = b, \tag{3.2}$$

then, the local discretization error satisfies

$$d_h = h^{-2}\delta^2 y_i - f(x_i, y_i) = h^{-2}\delta^2 y_i - y_i" = \frac{h^2}{12} y_i^{(4)} + \mathcal{O}(h^4), \tag{3.3}$$

where $y_i = y(x_i)$.

Thus, the global error satisfies the linearized equation $h^{-2}\delta^2 e_i = f_y e_i + \mathcal{O}(h^2)$. If we assume that the method is stable then this shows that the method is convergent of order 2.

This is also the key to deferred corrections.

Deferred correction algorithm:

1. Solve system (3.2) for $y_1 = \{y_i\}$.
2. Use y_1 to approximate the local discretization error (3.3) to order $\mathcal{O}(h^2)$ by, say, $dc_{1i} \sim \frac{h^2}{12} y_i^{(4)} + \mathcal{O}(h^4)$.
3. Solve $h^{-2}\delta^2 y_{2hi} = f(x_i, y2_{hi}) + dc_i,\ i = 1, ..., n-1 \quad y_{20} = 0, \quad y_{2n} = 0,$
4. Either use y_2 as a global error estimate or correct the $\mathcal{O}(h^2)$ solution to obtain an $\mathcal{O}(h^4)$ one by adding it to y_1.

Observe that we have obtained a higher order approximation by using (twice) the low order one, for which we have known convergence theory and efficient solution algorithms. This is similar to Richardson extrapolation, except that only one mesh is involved. See [45, 50] for more discussions and comparisons.

Another important implementation detail for nonlinear boundary value problems, or for stiff initial value problems, is that an intelligent implementation will couple tightly the solution of the $\mathcal{O}(h^2)$ operator with global error control and with the availability of accurate initial guesses from one Newton solve to the next.

It is quite interesting, as we will see later, how from this early emphasis on boundary value problems, followed the modern developments directed towards initial value problems and parallel implementations. Finally, an important observation is that if we have a good method to estimate the error, this estimate can be added to the discrete solution to improve it. Observe the important distinction between estimates and bounds.

3.4. **Iterated deferred corrections.** As a matter of fact, the last comment in the Section above and the availability of longer asymptotic expansions points to a more general recursive and adaptive algorithm, in which successive corrections are calculated incorporating higher and higher order terms of the asymptotic expansion, guided by the built in asymptotic error estimates. That is, we assume that a multiple term expansion exists and repeat the procedure of the previous section, incorporating at every step one more term of the expansion. It is possible to show that this bootstrapping procedure works and provides ever increasing order of accuracy, until the expansions do not lead to an improvement, hopefully predicted by an increase in the error estimate. Of course, this is connected to Fox's method of monitoring the behavior of the high order differences. Observe that these are asymptotic expansions, i.e., approximations (with sign!), and not rigorous a posteriori error bounds.

The explicit asymptotic expansions provide an straightforward way to generate the corrections, but sometimes they may not be available or be difficult to obtain. For instance, if derivatives of the solution appear in nonlinear terms, then the expansions become increasingly complicated. In a later Section we will see how a variant of deferred corrections, defect corrections, solves this problem. The detailed theory and different applications can be found in [45, 46, 48, 49, 50, 53, 52].

3.5. **Tools.** We have threaded lightly over the approximation of the asymptotic expansions, which in Fox's times would have involved the manual inspection of difference tables. Fortunately from early on, general, automatic algorithms were developed explicitly to facilitate this task, both in one and in multiple dimensions.

Those tools were related to Lagrangian interpolation, numerical differentiation and quadratures. The first papers in this line deal with the solution

of Vandermonde and transpose Vandermonde one dimensional systems [4, 5]. Vandermonde matrices are notoriously ill-conditioned (i.e., the polynomial basis is the problem and that is why orthogonal polynomials and other representations are used). However, the algorithm developed in these papers were capable of solving accurately systems of practical interest, as compared, say, to Gaussian elimination. N. Higham [30, 31] eventually proved rigorously the stability properties of the method.

Important additional points were that the actual Vandermonde matrix was never formed and the algorithm had $\mathcal{O}(n^2)$ complexity, as compared with the $\mathcal{O}(n^3)$ complexity of Gaussian elimination. In addition, this was the foundation for extensions to multiple dimensions. In p. 67 of [50] there is a listing of a Fortran program that implements a general weight generator for one dimensional problems on a general non-uniform mesh.

Given a linear differential operator

$$L(y)(x) = \Sigma_{j=0}^{m} a_{j+1} y^{(j)}(x) \frac{h^j}{j!},$$

we want to approximate it by a linear combination of function values:

$$M(y) = \Sigma_{s=1}^{m+1} w_s y(x_{s-r}),$$

where the points x_{s-r} are assumed to be distinct but not necessarily equispaced. A judicious use of Taylor's expansions leads to the solution of a Vandermonde system of equations to obtain the weights w_s. By choosing appropriately the parameters and data one can calculate the approximations required by different differential equations and deferred correction algorithms, including the case of non-uniform meshes and the fact that near the boundaries of the integration interval different approximations are required.

For dimensions higher than one the problem was approached in two different ways:

- Using tensor product approximations of one dimensional formulas (i.e., brick elements, requiring uniform grids); see the algorithms in [9, 61], which have a number of additional applications for fast, concise manipulation of tensor (or Kronecker) products of matrices.
- For approximation on simplexes (triangles in 2D), see [24]. The algorithm described there can be used to generate high order approximations to linear differential operators in any dimension, leading to discontinuous finite elements (never used for that purpose, as far as we know).

4. Zadunaisky's pseudo-problem and defect corrections

As we said above, the Argentine astronomer P. Zadunaisky was interested in carefully estimating the global error in the numerical integration of initial value problems for ordinary differential equations in his study of orbits of

comets. In [64] he presented an idea that eventually turned out to be an improvement and generalization of deferred corrections. Given a system of ODE's with initial conditions

$$y' = f(t, y), \quad y(a) = y_0$$

and a discretized version (Euler's method for simplicity in the explanation)

$$(y_{n+1} - y_n)/h = f(t_n, y_n),$$

the idea is to introduce a pseudo-problem with known solution, whose integration (by the same method) would yield estimates for the solution error.

The procedure involved solving the original problem and using the discrete solution to construct a polynomial approximant $P(t; y_h) \sim y(x)$ that then could also be used to approximate any derivatives. The pseudo-problem is then:

$$z' = f(t, z) + [P' - f(t, P)], \quad z(a) = y_0,$$

which has the exact solution $z = P$ by construction. Thus, if we solve this problem with the same original discrete method then the (heuristic) argument says that $z_i - P(t_i)$ will be an estimate of the error $y_i - y(t_i)$. This is predicated on choosing the approximant $P(t; y_h)$ appropriately.

This idea is reminiscent of deferred corrections but, as observed first by Bengt Lindberg [43], there are two crucial differences, namely: no asymptotic expansions are explicitly required and derivatives inside nonlinear functions can be approximated directly. However, asymptotic expansions can still be used to prove convergence of the enhanced order methods, just as in the case of deferred corrections. Of course, the term $P' - f(t, P)$ is an approximation to the local truncation error or defect, which bring us to the next development.

Zadunaisky presented a polished version of his results at the 1973 Dundee Conference on Numerical Analysis, where H. Stetter picked up the idea and converted it into a complete theory of defect corrections, very much as an extension of deferred corrections, not only for error estimates but also for improved solutions. The methods and theory followed closely Pereyra's developments (including the names), as one can see in [6], where there is a fairly comprehensive description of the state of the art circa 1983, with considerable elaboration on the interconnections between the different methods and many references.

Because of the advantages mentioned above for Zadunaisky's method, defect corrections became an important generalization, although the differences with deferred corrections are not too large and the practical lessons learned in the previous 20 years of development and applications are worth taking into

account. Observe in particular that the machinery used in deferred corrections (see Tools above) can also be applied to obtain high order polynomial approximants to the unknown function and its derivatives.

5. Spectral deferred corrections

In [11] the authors introduce a new class of deferred correction methods for initial value problems for ordinary differential equations (IVP's for ODE's) that they call spectral deferred corrections (SDC). They begin by converting the original ODE into the corresponding Picard integral equation and apply a deferred correction procedure in the integral formula, driven by the explicit or implicit Euler marching scheme. This approach leads to algorithms of essentially arbitrary order for both non-stiff and stiff problems. For non-stiff problems, algorithms with orders between 8 and 20 should be competitive with the best existing ones, while for stiff problems a simple adaptive implementation demonstrates performance comparable to state-of-the-art extrapolation codes. The authors of [32] connect spectral deferred corrections for linear problems with a pre-conditioned Neumann series expansion for the solution of the standard collocation discretization of the ODE. They use this observation to accelerate the convergence of SDC using the GMRES Krilov subspace method. For nonlinear problems, the GMRES acceleration is coupled with a linear implicit approach.

In [44], Minion considers multiple shooting combined with spectral deferred corrections in order to obtain a parallel high order algorithm for ordinary and partial differential equations. The parallel speedup and efficiency of the hybrid methods are analyzed and numerical results for ODEs and discretized PDEs are presented. Minion and his associates have published many papers on the subject and its variations for different applications.

6. Applications and software

6.1. **Ordinary differential equations.** The main emphasis in the 60's through 80's was in the use of deferred corrections for two-point boundary value problems (BVP) for systems of ordinary differential equations, since there was adequate software and methods available for the high order solution of initial value problems (IVP), at least pre-stiff problems.

As a matter of fact, if you would ask an engineer or scientist how she solved BPV the answer would be "by shooting". For one second order equation this is explained very simply:

$$y" = f(t, y), \quad y(0) = a, \quad y(1) = b.$$

Shooting algorithm.

(1) Choose an initial guess for the missing initial condition: $y'(0) = c$
(2) Integrate the IVP until $t = 1$.
(3) If $y(1) = b$ finish

(4) If $y(1) \neq b$, then adjust c and go to 2.

All the science goes into "adjust c". If everything works well, then this is a fairly straightforward procedure that only requires available IVP integrators.

However, there is a fundamental difficulty with the approach, even for linear problems. Namely, it may be impossible to integrate the equation from 0 to 1 for the given initial values because y goes to ∞ in finite time, i.e., the problem is very sensitive to small changes in the initial conditions. The extreme case occurs when changing the exact initial conditions that solve the boundary value problem by one bit produces a divergent solution. Observe that the BVP can be perfectly well conditioned while the IVP is not.

A palliative for this problem is multiple shooting [33, 8], which breaks the interval of integration into sub-intervals and shoots simultaneously in all the sub-intervals, enforcing continuity conditions at the joints. This is a hybrid between pure shooting and fully global discretization methods that consider a mesh and evolve the discrete solution from some guessed initial values, by solving a system of nonlinear algebraic equations provided by the finite difference approximation of the equations. There has been considerable interest in this approach in recent times from the point of view of parallel computing, since the integrations on the different intervals can be performed independently [39].

With the exception of [16], it was not until this century that interest in deferred/defect corrections was directed towards IVPs, mainly motivated by partial differential equations.

The public implementations of deferred corrections for global methods for BVPs, in the series of codes SYSSOL [40], PASVA3 [41, 51] and PASVA4 [42, 52] filled a need. They were well received and used for many years, especially when they were incorporated in most of the public repositories of the time, such as nanet, IMSL (as subroutine BVPFD), Harwell (as subroutine DD03AD), NAG (as subroutine D02RAF), etc. We acknowledge the work of I. Gladwell, Univ. of Manchester, that was instrumental in the documentation and testing of PASVA3 for its introduction in the Harwell, NAG and IMSL repositories. For some 2012 applications see [7, 28, 37, 56].

The first codes and experimentation were restricted to single second order equations [50]. This included the development of a general tool for generating deferred corrections automatically. SYSSOL, the first code for general first order nonlinear systems, was restricted to uniform meshes that could be refined automatically, but that proved to be left wanting for a number of challenging problems that contained high gradients in parts of the domain, such as boundary layers.

The advent of H. B. Keller's book [33] gave a great impulse to the extension of the ideas to systems of first order equations, since a complete theory was developed there, including discretizations on non-uniform meshes. The combination of these developments with those in [54], which established a

solid approach to the automatic choice of non-uniform meshes based in the concept of equidistribution of a monitor function (the local truncation error, in our approach), led to PASVA3 [41, 51], a code that implemented a variable order, variable step strategy for BVPs (what later on was called an h-p method). Good discussions of these and other related matters can be found in [3].

Later on, due to specific needs in an important application to seismic exploration [42, 52], this code was significantly extended to PASVA4, a code capable of solving problems with internal jump discontinuities at unknown locations and also determining additional nonlinear parameters. Curiously enough, the first real application of this code was that of Charlie Wilts at Caltech for modeling bubble memories [63].

Another faithful user was Professor Hors Ecker, University of Vienna [18], to whom we have a great debt, since he spent considerable effort a few years ago to recover an original copy of PASVA4 from a 1989 floppy disk (sic!) that facilitated its use in the work mentioned next. Although early on we thought that PASVA4 could be useful for solving bang-bang control problems, it was only fairly recently that Laura Perez et al, Univ. de Rio Cuarto, Argentina, used it for designing the optimal control of a hybrid vehicle [55].

From the first codes, a simple continuation option was introduced that could help overcome one of the main difficulties for nonlinear boundary value problems: the need to give a reasonable first approximation over the whole range of integration. Homotopy continuation could help guide the integration from a known solution to the desired one, by introducing a natural or artificial parameter and calculating a sequence of solutions. Of course, this could not work in general and it took several years until the work of Keller [34] and Doedel [17] provided impressive tools to calculate full bifurcation diagrams for complex non-linear problems. It is about this time that J. Cash, Imperial College, London, started his work on deferred corrections [12] that has continued strongly through the years [14, 13]. A good summary of recent work and much more can be found in the recent book [59].

A number of other codes emerged in the 70's and a good summary of the state of the art can be found in [15]. Most noticeable was COLSYS [1], a solver for high order systems using collocation with splines. It had a sophisticated mesh adaptation algorithm based on [54] and was able to solve stiff boundary layer and other complex problems (see [2]). We helped B. Russell in the early development of this code, so there is some commonality between it and the PASVAR family.

6.2. **Partial differential equations.** Early on L. Fox applied difference corrections to elliptic problems in two-dimensions [19, 20].
Pereyra also looked into this problem, including some nonlinear equations. In [45] there is a careful discussion of the theoretical difficulties that curved boundaries bring for high order methods, since it is non-trivial to assert the

existence of asymptotic expansions. Numerical results of linearized deferred corrections and comparisons with Richardson extrapolation are offered for the Dirichlet problem for mildly nonlinear 2D elliptic problems in regions with curved boundaries. As an important side issue, in this paper is introduced for the first time the notion of nested iterations and how to handle them efficiently. A theorem is proven that shows how to preserve the quadratic convergence of Newton's method when the corrections are only obtained approximately, i.e., by an iterative method, such as SOR. This result appeared years before the most quoted paper on the subject [10].
In [49] Pereyra discusses the n-dimensional problem on a convex region for a class of nonlinear elliptic equations and offers some additional results and comparisons using iterated deferred corrections. In both papers, the approximations near the boundaries are one-dimensional along grid lines, using only interior points. In the second paper, where derivatives may appear in the nonlinear part, there is a good discussion of the difficulties this brings to the derivation of asymptotic expansions, a good point for using defect corrections instead for this case. Comparisons with published results for the emerging technology of finite elements show a 17-fold speedup for deferred corrections to obtain similar accuracy.

This work lead to [53], where the authors study carefully the issue of solving Laplace's equation on curved regions by high order methods. There is shown that in general regions only a limited number of terms of an asymptotic expansion can be obtained, when using one-dimensional approximations for points near the boundaries. This was an unpublished result of H. O. Kreiss that the authors laboriously reconstructed. In fact, only two terms in an asymptotic expansion can be guaranteed, which in the L_2 norm leads to a best order of $\mathcal{O}(h^{5.5})$. It is conjectured there that in the infinity norm it would be possible to obtain a three terms expansion leading to a $\mathcal{O}(h^7)$ approximation.

It was not until this century that interest was rekindled on the application of deferred correction to problems in PDE's. A significant effort was started by Bertil Gustafsson and his collaborators at University of Uppsala, Sweden and very interesting results were obtained for the application of deferred corrections to obtain high order methods, both in space **and** time [27, 35, 26] for hyperbolic systems.

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